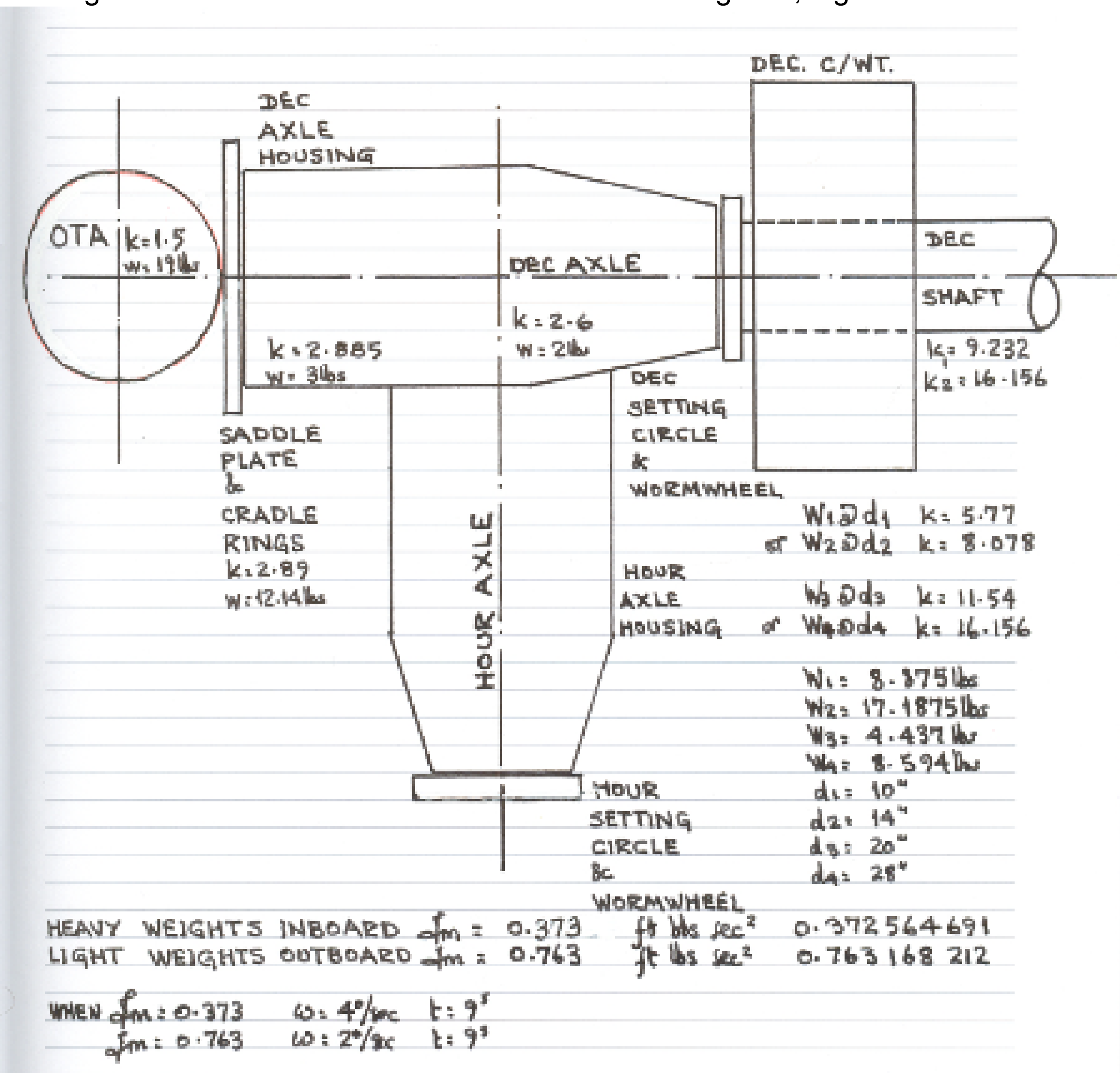
INERTIA & COUNTERBALANCING

Counterbalancing a German Equatorial entails placing counterweights along an extension to the Declination axle, the moment arm of which equals the moment arm of the telescope OTA including all its tube furniture, the saddle plate & cradle rings, and the Declination axle housing. Moments on the OTA side of the centre-of-gravity (cg) have to balance moments of the Dec shaft plus counterweights. The cg has to lie at the intersection of the hour axle and Declination axle.

I have been told, sometimes quite insistently, that it is better to put lighter counterweights further from the cg than heavier c/wts closer inboard.

This is a flawed notion, based on an inadequate understanding of dynamics. Yes you can produce the same moment by using half the weight at twice the arm length, but this ignores the angular momentum, and the inertia of the weights rotating about the hour axis. Trivial at sidereal tracking rate, significant at slew rate.



Lets look at a specific example, a Vixen GP-DX carrying a TEC140APO with a saddle and cradle, and two counterweights. In the first scenario there will be an

8.875lbsf weight @ 10-inches & a 17.1875lbsf weight @ 14-inches. These c/wts balance the OTA and saddle + cradle. In the second scenario there will be a 4.437lbsf weight @ 20-inches & a 8.594lbsf weight @ 28-inches, again balancing the OTA and saddle + cradle.

What happens when the GOTO stepper motor slews the OTA? The radius of gyration values for the OTA, saddle + cradle rings & Dec housing are unchanged, and hence the polar moment of inertia remains unaltered. It works out to be 0.067ftlbsec^2. The radius of gyration of the c/wts however are changed. Increase the moment arm by two and halve the weight, and you might be led to assume the radius of gyration would remain the same. But it doesn't, and for a very straightforward reason. The radius of gyration is the square root of the area moment of inertia divided by the cross sectional area, and the polar moment of inertia is the mass times the square of the radius of gyration. Halve the weight and double the moment arm and you double the polar moment of inertia.

From the above example you can see it increases from 0.373 to 0.763 ftlbsec^2.

If you input 0.373 into cell C20 in the spreadsheet I have prepared, downloadable from my <logbook> page <SLEWING TORQUE for GP-DX> you will see that for a 9sec acceleration time the OTA can be slewed at 4º/sec given the stepper motor pull out torque of 35mNm.

Input 0.763 into cell C20 in the spreadsheet and you will see that for the same 9sec acceleration time the slew rate drops to 2º/sec for the same pull out torque.

A spreadsheet, like any other form of computational programme is “black box”. You can see the outcome, but it doesn’t actually give you any insight. The ***only*** way to gain insight in mechanical engineering science is by using a mathematical model. That is why applied mathematics is so important, not only in mechanical engineering, but also amateur telescope making. The maxim is,

“If you don’t know it in numbers, then you don’t know it at all.”

So what does the mathematical model look like?

With the Vixen GP-DX, as with any other German Equatorially mounted telescope, we have a motor driven worm and wheel geared drive system, rotating the OTA about two mutually perpendicular axes, the hour axis and Declination axis. (It is ***not*** the RA axis - RA refers to how the equatorial coordinate system is measured about the celestial polar axis, i.e. Right Ascension - your mount may have an RA setting circle, but the polar axis rotates the OTA in, *“Hour Angle”* hence, “Hour Axis”).

The mathematical model has to take into account friction losses in the worm & wheel gear train. The motor itself usually has an inline gearhead which itself has losses, but the manufacturer’s data sheet specifies the output torque characteristic as measured. In the case of a stepper motor, what we need to know is the holding or detent torque, (the torque required to overturn the energized rotor through one step), and the pull out torque and maximum step frequency in pulses per second (pps or Hz).

The model also has to consider the inertia of all the moving parts of the mounting and telescope when it is rotated about the hour and Dec axes. This boils down to estimating the polar moments of inertia.

Inertia is something the layman thinks he understands, but in reality hasn't a clue about. We live and work in a 3D environment, and experience the passage of events caused by time's arrow. In everyday life we all experience time's arrow at the same rate. Hence Newton's three laws of motion adequately describe mechanical engineering phenomena.

We measure length, width and breadth of objects in three mutually perpendicular directions. Its called a reference frame. When you take into account an object's mass, or weight in a gravitational field, it's termed an "Inertial" reference frame. Measure along one axis of the reference frame, and you are working in 1D, or linear length L. Measure along two axes of the reference frame, and you are working in 2D, which defines area, which is L^2. Measure in three axes of the reference frame and you are working in 3D, which defines volume, which is L^3. Measure lengths in all three axes of the inertial reference frame plus the object's cross section about any single rotational axis and you have a measure of that object's area moment of inertia, and you are working in 4D, or L^4. The area moment of inertia is designated "I" and in the Imperial foot-pound-second system is measured in either in^4 or ft^4. For the size of objects we are dealing with, in^4 is the more appropriate unit.

Polar moment of inertia is related to mass moment of inertia, an object's ability to resist change in rotational speed about a specific axis. The larger the mass moment of inertia the smaller the angular acceleration about that axis for a given torque.

The polar moment of inertia is derived from the mass moment of inertia by:

$$J_m = \frac{W.k_0^2}{g}$$

where W is the object's weight in lbsf,
k_0 is the radius of gyration of the object
g is the force of gravity which is $32.2 ft/\sec^2$.

The Radius of Gyration of a mass about a given axis is the distance **d** from the axis. At this distance **d** an equivalent mass is thought of as a point mass. The moment of inertia of this point mass about the original axis is unchanged.

Hence:

$$k_0 = \sqrt{\frac{I}{A}}$$

where A cross sectional area in the plane of the rotation axis.

GERMAN EQUATORIAL INERTIA MODEL

WORM & WHEEL EFFICIENCY:

Worm & wheel efficiency: $\eta = \dfrac{\cos\vartheta - \mu\tan\lambda}{\cos\vartheta + \mu\cot\lambda}$

empirical formulae for coefficient of friction with mineral oil lubrication - stainless worm on bronze wheel:

$$\mu = 0.193 v^{-0277}$$ where ν is rubbing speed in ft/min

$$v = \frac{n\pi d}{12\cos\lambda}$$

$$\& \quad \tan\lambda = \frac{l}{\pi d}$$

where l = linear lead of worm screw
λ= helix lead angle
d = worm dia. [effective]
n = revs/min

$$\tan\vartheta = \tan\vartheta_x \cos\lambda$$

where ϑ_x = tooth profile semi-angle

As you can see the worm & wheel efficiency entails a tedious calculation starting with the tooth form. An ISO form worm tooth has a 30º semi-angle. The lead angle λ is calculated from the linear lead or pitch. Pitch is defined by teeth per inch. A metric worm pitch is simply defined by the crest to crest distance. If you have a 2mm pitch worm it is near as damn it 12 teeth per inch. Hence $l = \frac{1}{12}$ inches. The effective worm diameter is taken from the mid point between tooth crest and tooth root diameters.

The rubbing speed v is the distance a point on the worhwheel moves past a point on the worm in a given time. It is expressed in ft/min. The faster the rubbing speed the greater the efficiency η. Telescope worm and wheel drive systems are very inefficient because the rubbing speeds are so low, even on maximum slew. For the GP-DX the efficiency varies from 0.208 @ 1º/sec to 0.319 @ 8º/sec. At sidereal tracking rate efficiency drops to a mere 0.053.

ACCELERATION TORQUE @ WORM

I have provided a sample polar moment of inertia calculation <GP-DX Jm calc.pdf>. It is also a tedious process entailing each moving part, its shape, **d** value and weight. I have also repeated the example for two different sets of Dec c/wts, above.

Acceleration is defined by: $\alpha = \frac{(\omega_f - \omega_\circ)}{t}$

where α is acceleration in radians per second squared ($rads/\sec^2$), ω is the angular velocity in radians per second ($rads/\sec$), and t is the acceleration time in seconds. There are 2π radians in one revolution, so 1 rev/sec = 2π rads/sec. $\omega_f \,\&\, \omega_0$ designate the finish and start angular velocities. In this instance $\omega_0 = 0\, rads/\sec$.

Acceleration torque is defined by: $T_o = \frac{J_m.\alpha}{120.\eta}$

where 120 is the GP-DX worm & wheel ratio.

This defines the torque required to turn the hour axis at the worm, but we also need to know if the stepper motor can actually provide sufficient torque to overcome the inertia of all the moving parts being turned about the hour axis (the same calculation is required for the Dec axis which has to be undertaken separately and in addition to the hour axis calculation).

A stepper motor will stall if the required torque exceeds the pull out torque. It will also stall when the running frequency exceeds the commutation frequency <http://en.wikipedia.org/wiki/Stepper_motor>.

The GP-DX MT-1 stepper motor unit is a Nippon PF42-48G 120:1 inline gearhead with a maximum step rate of 310 pps. The pull out torque at the gearhead is about 35mNm or 0.31 inlbs. The polar moment of inertia of the rotor is $J_0 = 9.441 \times 10^{-7}\, ftlbs \sec^2$.

What we need to know is the start up frequency which is given by:

$$f_1 = \frac{f_s}{\sqrt{1 + \frac{J_L}{J'_O}}}$$

where f_s is the maximum start up frequency (320 pps) and $J_L = J_m.g$ & $J'_o = J_o.g$

from which the stepper start up torque at the rotor is given by:

$$T_a = \frac{\eta\,(J'_O + J_L).\pi.\vartheta_s.(f_2 - f_1)}{180.t}$$

where ϑ_s is the step angle (7º.5) &
f_2 is the stepper pulse rate at slew speed.

$$f_2 = \frac{\omega.120}{\vartheta_s}$$

The torque at the worm is given by:

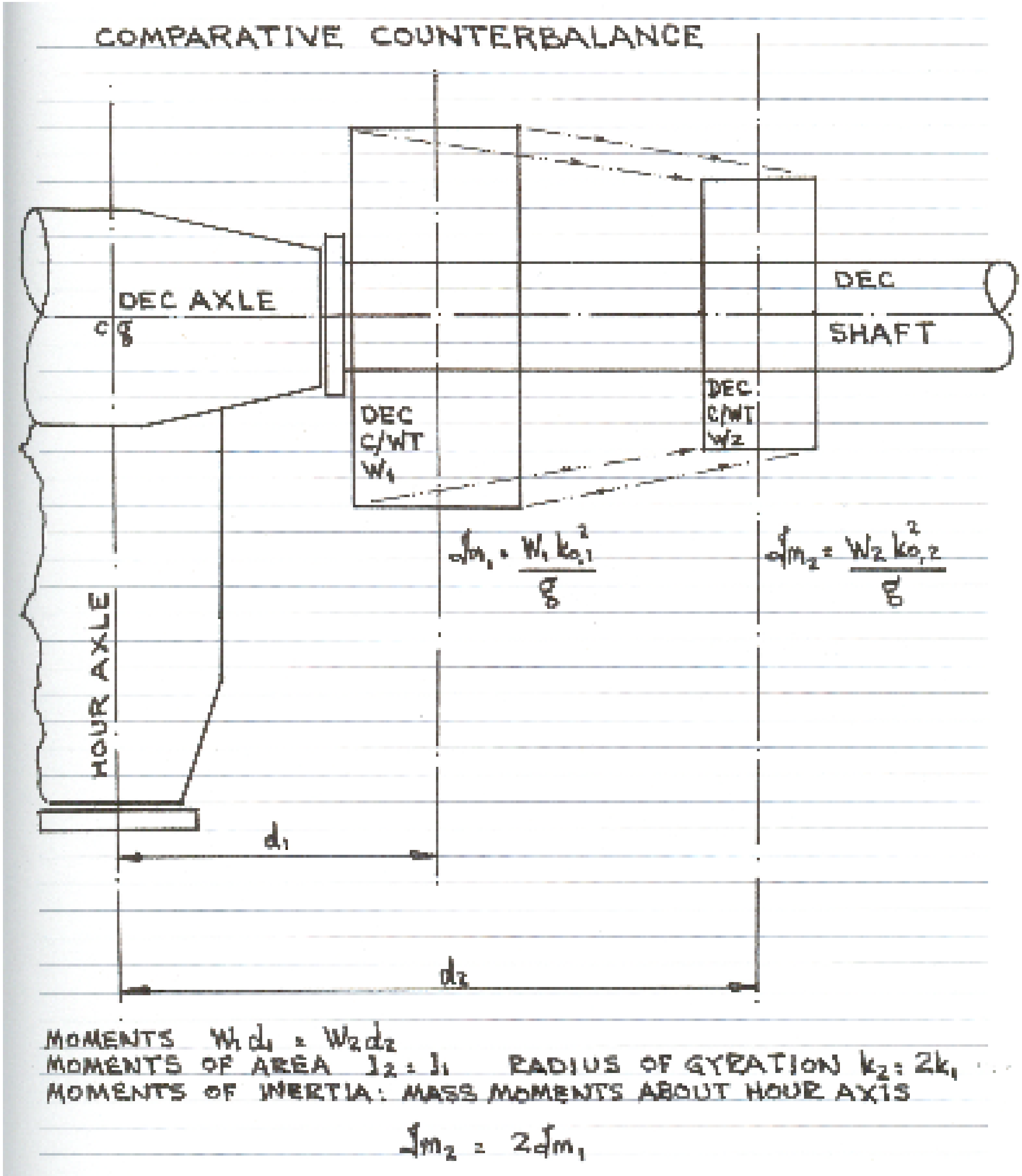
$$T_{aw} = \frac{T_a}{120}$$

where 120 is the inline gearhead reduction ratio.

COMPARISON of COUNTERBALANCE SCENARIOS

In comparing slewing torques between c/wts placed at twice the distance from the cg, weighing half as much, the only thing that varies are the J_m values of the c/wts.

Taking J_{m1} & J_{m2} as the polar moments of inertia, c/wts inboard & outboard, since the outboard c/wts are half the weight of the inboard c/wts and are placed at twice the distance from the cg $J_{L1} = J_{m1}.g \propto W_1.k_{0,1}^2$ & $J_{L2} = J_{m2}.g \propto W_2.k_{0,2}^2$ and since $k_0 = 0.577d$ the ratio between J_{L1} & J_{L2} is 2:1 hence $J_{L1} = 0.5J_{L2}$.



But the maximum pull out torque is fixed, and because J'_0 & $f_1 << J_L$ & f_2 the stepper pulse rate at slew speed has to be halved, hence $f_{2,1} = 2f_{2,2}$.
QED
i.e.
Place the c/wts at twice the distance from the cg, having half the weight, and you halve the maximum attainable slew speed.

CONCLUSION

If you find this result counterintuitive it is because you think in terms of statics. You are not alone. In fact you are amongst illustrious company. I was once reliably informed by one professional astronomer with a BSc in Astronomy & Mathematics from St. Andrews and an MSc in Astrophysics from Aston, that it is better to place a lighter weight further from the cg. The astronomer in question has worked for over 25 years in Australia and is famous for his comet hunting prowess, but never allow ignorance of the facts prevent you from expressing an "*expert*" opinion, even if it is in a field about which you know nothing.

A stepper motor, having a specific pull out torque for a particular communtation controller, will be doing half the work accelerating double the mass at half the distance, so given the amount of work in a given time is fixed, it can accelerate that doubled mass to twice the speed in the same time.

Still not convinced? Then take the argument to *reductio ad absurdum*. Extend the Dec c/wt along a massless shaft to infinity minus one, and reduce the weight to almost nothing. It will take almost infinite energy to start rotating the miniscule weight about the hour axis.

This is the primary reason why having a lighter weight on a longer Dec shaft is a bad idea. But there are three other mechanical engineering reasons why it is a bad idea.

- 1 The natural frequency of the mounting & OTA. A longer Dec shaft has a much lower natural frequency, given that it is the same diameter.
- 2 The damping time, the longer shaft resonates for longer.
- 3 Deflection of the Dec shaft under a point end load, given by:

$$\delta = \frac{W.L^3}{3.E.I}$$

where W is the load in lbs
L is the cantilever length
E is Young's modulus of elasticity
$E = 28 \times 10^6$ lbsf/sq.in. for mild steel
I = area moment of inertia $I = \frac{\pi.d^4}{64}$
where d = Dec shaft diamter

Because deflection δ is proportional to the cube of the cantilever length L for a given weight W if you halve the weight and double the cantilever length the deflection quadruples.

COMMENT

In the real world you may have noticed how expensive Dec c/wts have become. Roughly £10 / lbf which in my book is a rip off. After all we're only talking black mild steel, cut to length, bored out, and painted or powder coated.

If like me, you refuse to throw hard earned money at a middle man just for a steel c/wt, may I recommend two cheap alternatives.

• York standard Cast Iron Disc weight, 2.5kg (1lb:2oz) @ £16
<http://www.amazon.co.uk/gp/product/B0015UP914/ref=twister_dp_update?ie=UTF8&childASIN=B0015UR66U>,
• TESCO chrome plated York 15kg Dumbell Sets for £40
<http://direct.tesco.com/q/R.200-3674.aspx>

Cast iron weights can be secured using a jubilee clip.

Alternatively cast your own out of lead. Buy the lead from a scrapyard, make your own mould out of plywood lined with plaster of Paris dried in an oven to drive out all the moisture (very important), and melt the lead in an old saucepan. Alternatively the lead could be poured into an old cake or biscuit tin. Bore out the ID afterwards using a spade drill, or securely fasten a wooden mandrel in the centre of the mould. The wood will char, but the finished weight can be dressed afterwards with a wire brush.

POST SCRIPT

There is a completely misguided notion that modern GOTO controllers will cater for out-of-balance moments. It matters not whether a GOTO system is open or closed loop, if the telescope is out-of-balance, resonant vibration will increase. If the stepper motor has to be driven at or through its cogging frequency which coincides with, or is a harmonic of, the natural frequency of the equatorial, the drive and mount will resonate at that frequency. Typically it is a few Hz. A powerful DC Servo drive can cope with slightly greater out-of-balance moments than an equivalent torque rated stepper drive. But if the telescope is well out of balance, the maximum slew rate will drop, or the stepper motor may stall.

It is easy to balance a telescope where all the parts are placed in the plane of the Dec axis. If you are obliged to place tube furniture out of the plane of the Dec axis, I recommend you consult Bob Cox's article in Sky & Telescope on balancing a Newtonian telescope, (it applies to any type of telescope. Tube furniture can be placed in the plane of both hour and Dec axes on a fork mount).

Ref: Sky & Telescope, Gleanings for ATM's, Robert E. Cox, Nov.1958 pp47-50 & Dec.1958 pp107-109.
From an era when amateur astronomers were expected to be well educated, numerate, and able mechanical engineers, and S&T was a subscription only magazine, not the dumbed down high street popularist mag of today. How I miss Bob Cox. Old School is the ***only*** school worth paying the slightest attention.

The articles may be downloaded from my logbook page:
<S&T article_Newtonian Balance.zip>.

Remember,

"If you don't know it in numbers, then you don't know it at all."

INERTIA & COUNTERBALANCE PROOF

To prove: Slew rate is proportional to acceleration torque irrespective of worm drive efficiency

Given: Telescope equatorial tube assembly (OTA) + tube furniture
Saddle plate & OTA cradle
Declination axle and axle housing & worm & wheel drive
Hour axle & worm & wheel drive
Declination axle extension shaft
Declination counterbalance weight(s)

Proof:

EFFICIENCY of WORM DRIVE:

$$\eta = \frac{\cos\vartheta - \mu\tan\lambda}{\cos\vartheta + \mu\cot\lambda} \qquad (1)$$

empirical formulae for coefficient of friction, mineral oil lubrication - stainless worm on bronze wheel:

$$\mu = 0.193\nu^{-0277} \qquad (2)$$

where ν is rubbing speed in ft/min

rubbing speed at slewing rate:

$$\nu = \frac{n\pi d}{12\cos\lambda}$$

$$\& \quad \tan\lambda = \frac{l}{\pi d}$$

where
l = linear lead of worm screw
λ= helix lead angle
d = worm dia. [effective]
n = worm rate revs/min

$$\tan\vartheta = \tan\vartheta_x\cos\lambda \qquad (3)$$

where ϑ_x = tooth form semi-angle

SLEW ACCELERATION:

$$\alpha = \frac{\left(\omega_f - \omega_\circ\right)}{t} \qquad (4)$$

where
ω_0 = start up speed rads/sec
ω_f = finish speed rads/sec
t = acceleration time secs

POLAR MOMENT of INERTIA:

$$J_m = \frac{W k_0^2}{g} \qquad (5)$$

where
W = weight of Dec c'wt
k_0 = area moment of inertia
g = acceleration due to gravity

Putting $J_{m1} = \frac{W_1.k_{0,1}^2}{g}$ for counterweight 1

$J_{m2} = \frac{W_2.k_{0,2}^2}{g}$ for counterweight 2

& $k_{0,1} = 0.577.d_1$ area moment of inertia c'wt 1

$k_{0,2} = 0.577.d_2$ area moment of inertia c'wt 2

given $W_1 = 2W_2$ & $d_2 = 2d_1$ $J_{m2} = 2J_{m1}$ ____________ (6)

SLEW ACCELERATION TORQUE:

$$T_{\alpha 1} = \frac{\eta_1 (J_O + J_{L1}).\pi.\vartheta_s.(f_{2,1} - f_{1,1})}{180.t}$$
$$T_{\alpha 2} = \frac{\eta_2 (J_O + J_{L2}).\pi.\vartheta_s.(f_{2,2} - f_{1,2})}{180.t} \quad \text{(7)}$$

where
- $T_{\alpha 1}$ is torque for c'wt 1
- $T_{\alpha 2}$ is torquw for c'wt 2
- η_1 is worm efficiency c'wt 1
- η_2 is worm efficiency c'wt 2
- J_0 is rotor polar inertia
- J_{L1} is polar inertia c'wt 1
- J_{L2} is polar inertia c'wt 2
- $f_{2,1}$ is slew frequency c'wt 1
- $f_{1,1}$ is start frequency c'wt 1
- $f_{2,2}$ is slew frequency c.wt 2
- $f_{1,2}$ is start frequency c'wt 2
- ϑ_s = step angle - degrees
- t is acceleration time - secs

$$J_{L1} = J_{m1}.g$$
$$J_{L2} = J_{m2}.g$$

but $J_{m2} = 2J_{m1}$ $\therefore$ $J_{L2} = 2J_{L1}$

but the acceleration torque is fixed

$\therefore$ $T_{\alpha 1} = T_{\alpha 2}$ & $f_{2,1} = 2f_{2,2}$

startup frequency $f_1 = \frac{f_s}{\sqrt{1 + \frac{J_L}{J_0}}}$ ____________ (8)

where f_s is the max start up frequency

and because $J_L >> J_0$, $f_1 << f_2$

SLEW ACCELERATION TORQUE RATIO:

$$\frac{T_{\alpha 1}}{T_{\alpha 2}} = \frac{\eta_1(J_0 + J_{L1})(f_{2,1} - f_{1,1})}{\eta_2(J_0 + J_{L2})(f_{2,2} - f_{1,2})} \cong \frac{\eta_1.J_{L1}.f_{2,1}}{\eta_2.J_{L2}.f_{2,2}}$$

but $J_{L2} = 2J_{L1}$ & $f_{2,1} = 2f_{2,2}$ $\therefore$ $\frac{T_{\alpha 1}}{T_{\alpha 2}} = \frac{\eta_1}{\eta_2}$ ____________ (9)

from (1) ϑ & λ are constants $\quad \therefore \frac{T_{\alpha 1}}{T_{\alpha 2}} \propto \frac{\mu_1}{\mu_2}$ ______________________(10)

where μ_1 is coefficient of friction c'wt 1
μ_2 is coefficient of friction c'wt 2

from (2) $\quad \frac{\mu_1}{\mu_2} = \frac{{\nu_1}^{-0.277}}{{\nu_2}^{-0.277}}$

but $\nu_1 = 2\nu_2$ $\quad \therefore \frac{\mu_1}{\mu_2} = \left(\frac{1}{2}\right)^{-0.277} = 1.212$ ________________(11)

$\therefore \mu_1 = 1.212\mu_2$

but

$$\frac{\eta_1}{\eta_2} = \frac{(\cos\vartheta - \mu_1 \tan\lambda)(\cos\vartheta + \mu_2 \cot\lambda)}{(\cos\vartheta - \mu_2 \tan\lambda)(\cos\vartheta + \mu_1 \cot\lambda)}$$

$$= \frac{\cos^2\vartheta - \mu_1 \cos\vartheta \tan\lambda + \mu_2 \cos\vartheta \cot\lambda - \mu_1\mu_2}{\cos^2\vartheta - \mu_2 \cos\vartheta \tan\lambda + \mu_1 \cos\vartheta \cot\lambda - \mu_1\mu_2}$$

___________________________(12)

for ISO pitch worm $\quad \vartheta_x = 30^\circ$
for 12tpi x 5/8" worm $\quad \lambda = 2^\circ.43025$

subst in (11)

$$\frac{\eta_1}{\eta_2} = \frac{0.75 - 0.036764\mu_1 + 16.839794\mu_1 - 0.8250825\mu_1^2}{0.75 - 0.030333\mu_1 + 20.409831\mu_1 - 0.8250825\mu_1^2}$$

$$= \frac{0.75 + 16.80299\mu_1 - 0.8250825\mu_1^2}{0.75 + 20.379498\mu_1 - 0.8250825\mu_1^2}$$

& taking roots $\quad \frac{\eta_1}{\eta_2} \propto \frac{-0.04453751}{-0.03674702}\mu_1 \propto 1.212\mu_1$ ____(13)

subst in (10) $\quad T_{\alpha 1}\eta_2 = T_{\alpha 2}\eta_1$
$\eta_2\mu_1 = \eta_1\mu_2$ ______________________(14)

from (9) comparing (11) & (13) $\quad \frac{\eta_1}{\eta_2} = \frac{\mu_1}{\mu_2}$ ________________________(15)

subst in (14) $\quad T_{\alpha 1} = T_{\alpha 2}$ QED

CONCLUSION:

Use a Declination counterbalance weighing half as much at twice the distance from the cg along the Declination shaft extension, and the maximum slew speed halves, irrespective of the change of worm drive efficiency. This is a consequence of the maximum acceleration torque remaining constant for any given acceleration time.