

DERIVATION of FRESNEL'S EQUATION from YOUNG'S PRINCIPLES

First case - incident light polarized in plane of incidence, displacement perpendicular to plane of incidence.

Let amplitude of displacement at the boundary interface be, f for the incident ray, g for the reflected ray, and h for the refracted ray.

Energy propagated across cross-section of incident, reflected and refracted beams are proportional to:

$$c_1\rho_1f^2, \quad c_1\rho_1g^2, \quad c_2\rho_2h^2$$

where c_1, c_2 , denote the velocities of light in the respective media, and ρ_1, ρ_2 the densities of the aether in the two media.

The cross-sections of the beams which meet the interface in unit area are: $\cos i$, $\cos i$, $\cos r$ respectively.

The principle of conservation of energy therefore gives:

$$c_1\rho_1 \cos i.f^2 = c_1\rho_1 \cos i.g^2 + c_2\rho_2 \cos r.h^2 \quad \text{-----} \quad (1)$$

The equation of continuity of displacement at the interface is: $f + g = h$ ----- (2)

& from Snell's law: $\frac{\sin^2 r}{\sin^2 i} = \frac{c_2^2}{c_1^2} = \frac{\rho_1}{\rho_2}$ ----- (3)

Eliminating h in (1) from (2) and substituting (3) we have:

$$c_1\rho_1 \cos i.f^2 = c_1\rho_1 \cos i.g^2 + c_2\rho_2 \cos r.(f + g)^2$$

$$c_1\rho_1 \cos i.f^2 = c_1\rho_1 \cos i.g^2 + c_2\rho_2 \cos r.f^2 + c_2\rho_2 \cos r.g^2 + c_2\rho_2 \cos r.2fg$$

$$c_1\rho_1 \cos i.(f^2 - g^2) = c_2\rho_2 \cos r.(f^2 + g^2) + c_2\rho_2 \cos r.2fg$$

$$\div c_2\rho_2 \cos r.2fg$$

$$\frac{c_1\rho_1 \cos i.(f^2 - g^2)}{c_2\rho_2 \cos r.2fg} = \frac{c_2\rho_2 \cos r.(f^2 + g^2)}{c_2\rho_2 \cos r.2fg} + 1$$

subst (3)

$$\frac{\sin i \sin^2 r \cos i (f^2 - g^2)}{\sin r \sin^2 i \cos r 2fg} = \frac{c_2\rho_2 \cos r (f^2 + g^2)}{c_2\rho_2 \cos r.2fg} + 1$$

$$\frac{\sin r \cos i (f^2 - g^2)}{\sin i \cos r 2fg} = \frac{c_2\rho_2 \cos r (f^2 + g^2)}{c_2\rho_2 \cos r.2fg} + 1$$

$\times 2fg$

$$\frac{\sin r \cdot \cos i}{\sin i \cdot \cos r} (f^2 - g^2) = (f^2 + g^2) + 2fg$$

factorise $(f^2 - g^2)$ & $(f^2 + g^2)$

$$\frac{\sin r \cdot \cos i}{\sin i \cdot \cos r} (f + g)(f - g) = (f + g)(f + g) - 2fg + 2fg$$

$$\frac{\tan r}{\tan i} = \frac{\sin r \cdot \cos i}{\sin i \cdot \cos r} = \frac{(f + g)(f + g)}{(f + g)(f - g)} = \frac{(f + g)}{(f - g)}$$

Fresnel equation 1:

$$(f - g) \sin r \cdot \cos i = (f + g) \sin i \cdot \cos r$$

$$f \cdot \sin r \cdot \cos i - g \cdot \sin r \cdot \cos i = f \cdot \sin i \cdot \cos r + g \cdot \sin i \cdot \cos r$$

$$f \cdot \sin r \cdot \cos i - f \cdot \sin i \cdot \cos r = g \cdot \sin i \cdot \cos r + g \cdot \sin r \cdot \cos i$$

$$f(\sin r \cdot \cos i - \sin i \cdot \cos r) = g(\sin i \cdot \cos r + \sin r \cdot \cos i)$$

$$\frac{f}{g} = \frac{(\sin i \cdot \cos r + \sin r \cdot \cos i)}{(\sin r \cdot \cos i - \sin i \cdot \cos r)} = \frac{\sin(i + r)}{\sin(r - i)}$$

$$\frac{g}{f} = \frac{\sin(r - i)}{\sin(r + i)}$$

Fresnel equation 2:

$$(f - g) \tan r = (f + g) \tan i$$

$$f \cdot \tan r - g \cdot \tan r = f \cdot \tan i + g \cdot \tan i$$

$$f \cdot \tan r - f \cdot \tan i = g \cdot \tan i + g \cdot \tan r$$

$$f(\tan r - \tan i) = g(\tan i + \tan r)$$

$$\frac{f}{g} = \frac{(\tan i + \tan r)}{(\tan r - \tan i)} = \frac{\tan(i + r)(1 - \tan i \cdot \tan r)}{\tan(r - i)(1 + \tan r \cdot \tan i)} = \frac{\tan(i + r)(1 - \tan i \cdot \tan r)}{-\tan(i - r)(1 - \tan i \cdot \tan r)} = \frac{\tan(r + i)}{\tan(r - i)}$$

$$\frac{g}{f} = \frac{\tan(r - i)}{\tan(r + i)}$$

from Fresnel equation 1

$$\frac{\sin(r-i)}{\sin(r+i)} = \frac{\sin r \cos i - \cos r \sin i}{\sin r \cos i + \cos r \sin i}$$

Snell's Law:

$$n_1 \sin i = n_2 \sin r$$

$$\sin r = \frac{n_1}{n_2} \sin i \quad \text{-----} \quad (4)$$

subst (4)

$$\begin{aligned} \frac{\sin(r-i)}{\sin(r+i)} &= \frac{\frac{n_1}{n_2} \sin i \cos i - \sin i \cos r}{\frac{n_1}{n_2} \sin i \cos i + \sin i \cos r} \\ &= \frac{\sin i \left(\frac{n_1}{n_2} \cos i - \cos r \right)}{\sin i \left(\frac{n_1}{n_2} \cos i + \cos r \right)} \end{aligned}$$

$\times n_2$

$$\frac{\sin(r-i)}{\sin(r+i)} = \frac{n_1 \cos i - n_2 \cos r}{n_1 \cos i + n_2 \cos r} \quad \text{-----} \quad (5)$$

$$\text{subst } \cos r = \sqrt{1 - \sin^2 r} = \sqrt{1 - \left(\frac{n_1}{n_2} \sin i \right)^2} \quad \text{-----} \quad (6)$$

$$\frac{\sin(r-i)}{\sin(r+i)} = \frac{n_1 \cos i - n_2 \sqrt{1 - \left(\frac{n_1}{n_2} \sin^2 i \right)^2}}{n_1 \cos i + n_2 \sqrt{1 - \left(\frac{n_1}{n_2} \sin^2 i \right)^2}}$$

(Fresnel equation for polarization perpendicular to incident plane: s-polarized)

from Fresnel equation 2

$$\begin{aligned} \frac{\tan(r-i)}{\tan(r+i)} &= \frac{\sin(r-i)}{\cos(r-i)} \times \frac{\cos(r+i)}{\sin(r+i)} \\ &= \frac{(\sin r \cdot \cos i - \cos r \cdot \sin i)(\cos r \cdot \cos i - \sin r \cdot \sin i)}{(\cos r \cdot \cos i + \sin r \cdot \sin i)(\sin r \cdot \cos i - \cos r \cdot \sin i)} \\ &= \frac{\sin r \cdot \cos r \cdot \cos^2 i + \sin r \cdot \cos r \cdot \sin^2 i - \cos^2 r \cdot \sin i \cdot \cos i - \sin^2 r \cdot \sin i \cdot \cos i}{\sin r \cdot \cos r \cdot \cos^2 i + \sin r \cdot \cos r \cdot \sin^2 i + \sin^2 r \cdot \sin i \cdot \cos i + \cos^2 r \cdot \sin i \cdot \cos i} \\ &= \frac{\sin r \cdot \cos r (\cos^2 i + \sin^2 i) - \sin i \cdot \cos i (\cos^2 r + \sin^2 r)}{\sin r \cdot \cos r (\cos^2 i + \sin^2 i) + \sin i \cdot \cos i (\cos^2 r + \sin^2 r)} \\ &= \frac{\sin r \cdot \cos r - \sin i \cdot \cos i}{\sin r \cdot \cos r + \sin i \cdot \cos i} \end{aligned}$$

$\div \sin i$

$$\begin{aligned} &= \frac{\frac{\sin r \cdot \cos r}{\sin i} - \cos i}{\frac{\sin r \cdot \cos r}{\sin i} + \cos i} \end{aligned}$$

Snell's Law:

$$n_1 \sin i = n_2 \sin r$$

$$\frac{\sin r}{\sin i} = \frac{n_1}{n_2} \quad \text{----- (7)}$$

subst (7)

$$\begin{aligned} &= \frac{\frac{n_1}{n_2} \cos r - \cos i}{\frac{n_1}{n_2} \cos r + \cos i} \end{aligned}$$

$$\times n_2 = \frac{n_1 \cos r - n_2 \cos i}{n_1 \cos r + n_2 \cos i}$$

subst (6)

$$\frac{\tan(r-i)}{\tan(r+i)} = \frac{n_1 \sqrt{1 - \left(\frac{n_1 \sin i}{n_2}\right)^2} - n_2 \cos i}{n_1 \sqrt{1 - \left(\frac{n_1 \sin i}{n_2}\right)^2} + n_2 \cos i}$$

(Fresnel equation for polarization parallel to incident plane: p-polarized)