A report upon the analysis of the telescopic resolution of double stars of unequal brightness

Abstract:

The substance of this paper deals with the analysis of visual estimates of the angular separations of double stars of unequal brightness at the threshold of telescopic resolution. The measures are both those obtained in my own work which supplement those published by the professional double star observer, Lewis, in the ‘Observatory’ magazine of the Royal Astronomical Society in 1914.

The data has been analyzed using the method of least squares, and a law generated by fitting the means to a logarithmic curve. The goodness of fit has been determined by calculating the correllation coeficient.

The nature of the derived relationship is explored and investigated from the standpoint of what is known theoretically about the spurious image of a star, known as the ‘Point Spread Function.’

The data and the empirical and theoretical relationships are modelled in terms of those factors which effect telescopic resolution, and graphs and a nomogram are provided to illustrate the nature of the relationships and assist in solving the limiting equation governing the telescopic resolution of unequal binaries.

Telescopic Resolution of Unequal Binaries

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This paper considers the historical aspects to the various aperture dependent resolution limits commonly applied to telescope objectives figured to provide diffraction limited images, and how such limits can be modified to predict the diffraction limited aperture required to resolve pairs of unequal brightnesses.

Fraunhofer diffraction limits the objective resolution of any optical system.[1] How one defines that limit is a difficult problem for which there can be no definite answer. Resolution is dependent not only on the aperture, but the nature of the image.[2] Fortunately when considering the resolution of astronomical telescopes, matters become somewhat simplified, and when considering the telescopic resolution of double stars, even more so. And yet, the derivation of a law governing the telescopic resolution of unequal binaries has remained an unsolved problem for over one hundred and fifty years.

Fraunhofer diffraction by a circular aperture, including apertures possessing a central obstruction, was first investigated theoretically by George Biddell Airy in a paper presented to the Cambridge Philosophical Society on November 24th. 1834.[3] The nature of the equation which Airy derived had been known for some time, in fact since the theoretical work pioneered by Fraunhofer into the appearance of rectangular apertures used in the spectroscopes he devised for measuring the dispersion of optical glasses.[4]

The difficulty was applying it to a circular aperture. Airy was not the first to attempt a solution. The German mathematician Schward arrived at a viable solution using as an approximation, a polygonal aperture possessing 180 sides.[5]

The expression defining the spurious disc formed by a point source at infinity, at the focal plane of the objective takes the form:

$$4a^2 \sin\frac{2\pi}{\lambda}(vt - f - A)\int_{\omega}\sqrt{(1-\omega^2)}\,\cos n\omega . d\omega$$

where $n = \frac{2\pi}{\lambda}.\frac{ba}{f}$

&

where a is the objective radius
b is the lateral distance of a point normal to the optical axis from the focus
$\omega = x.a^{-1}$, x being an ordinate in the plane of the objective
λ = wavelength of light
vt = phase separation (light path distance between entrance pupil and image plane.)
A = arbitary constant
f = focal length

The Jacobean elliptic integral belongs to a class of linear differential equations known as Bessel functions of the first kind. It cannot be integrated by traditional methods which treat integration as the reverse of differentiation. It has to be solved numerically by expanding $\cos n\omega$ using a Taylor series expansion and integrated by parts. Airy was the first to succeed in doing so, and although the method was tedious by the contemporary methods available to him, the Bessel functions may nowadays be readily solved using a programmeable computer.[6]

Having demonstrated that the wave nature of light produced the phenomenon of Fraunhofer diffraction, Airy was able to model the distribution of light in the spurious disc of a point source formed by a circular aperture. He also showed how an axial sub-aperture stop modified the diameter and the distribution of light in the diffraction pattern, in an unexpected and complex way.

He also proved the angular diameter of the spurious disc was inversely proportional to the objective aperture, and had no dependence on either the magnification or focal length as had, until that time, been assumed. (*That is why throughout the C17th & C18th a telescope's power was defined by either its focal length or its focal ratio, rather than its aperture*).[7]

Because a diffraction limited objective forms a spurious disc (the so called 'Airy' disc) instead of a geometric point, when imaging a star (or any point source); a disc whose angular diameter is inversely proportional to the objective aperture; it became immediately evident to those who were able to comprehend the implications of Airy's work, that telescopic resolution was dependent on aperture alone. Magnification or focal length had nothing to do with resolution, except inasmuch as the image needed magnifying a certain amount before the eye could resolve the detail in it.

The snag was of course not many double star observers did understand Airy's paper. However one who did was his contemporary, John Herschel. Both Airy and Herschel had been Senior Wranglers and had both graduated from Cambridge having attained firsts in the mathematical tripos. Both ranked amongst the most brilliant mathematicians of their day. So brilliant that either could have derived and solved the integral. In fact Airy, in referring to his solution tacitly admits it was something he considered doing only because it had become a source of iritation to him. Airy was above all else a most practical man, and here was a quantifiable phenomenon consequent upon the undulatory nature of light, whose practical effects recurred perpetually and for which, at that time, there was no complete investigation.

In his ' Outlines of Astronomy,' John Herschel [8] describes the Airy disc and how it may be modified by a central stop to improve the objective's resolving power upon double stars. Herschel recommended the observer place an opaque disc, *"centrally before the object-glass, having a diameter from a sixth to a fifth of that of the object-glass."*

The reason Herschel made this suggestion was because he understood the effect a central obstruction had on the Airy disc. By reducing the diameter of the central light and putting more light into the rings, it should, at least in theory, be possible to improve the resolution of the objective.

The famous double star observer, William Rutter Dawes [9] took up John Herschel's suggestion and described the effects as ... *"increasing the* ***separating*** *power of the telescope, but at the same time increasing both the number and the brightness of the rings round the brighter stars ... The small companions of rather bright stars are often hidden by them; and the discs of nearly equal stars are apt to be elongated by the rings passing through them. I have therefore seldom used this expedient, though in some instances it is undoubtedly advantageous."*

Dawes summarised his lifelong experience with double stars in a Memoir of the Royal Astronomical Society, published in 1865.[10] In it he concluded by stating his limit for an unobstructed aperture. This now famous criterion was based upon the resolution of equal pairs of approximately the sixth magnitude, seen in a six inch refractor. In so defining his results he was aware that the resolvability of equal bright and equal faint pairs might differ. Yet he, in my opinion correctly, stated that he did not think the distinction a very great one.

His argument, which Airy's theory supports, is that the increased magnification a bright star might allow does not alter the disc-first interspace ratio. The angular size remains the same. However his argument is not entirely valid. The perceived diameter of the Airy disc does depend on the brightness of the star. Airy [3] quotes two boundary conditions to the edge of the disc centre by way of illustration. The radius of the spurious disc of a faint star, where less than half the intensity of the central light makes no impression on the eye ($S = 1.17/a$), and that of a bright star where light of one tenth the intensity of the central light is sensible ($S = 1.97/a$). ('a' is the objective radius in inches). Bright equal pairs might not be so easily resolved as slightly fainter equal pairs. Dawes was however clear on one vital point. His limit applied only to equal pairs. Unequal pairs would require a bigger aperture, though how much bigger he did not speculate.

So how valid is Dawes' limit in practice? Dawes' limit is empirical, based upon practical double star observation, and because it is a purely empirical limit, its validity is assured only for small refractors and Dawes' "Eagle Eye." *(Dawes' renowned acuity did not extend beyond the telescope. He was so myopic he is rumoured to have passed his wife in the street without recognizing her! Whether by omission or comission is not mentioned, though evidently she was no 'Madam Gamma.')*

There are two commonly employed resolution limits based upon diffraction theory. They are derived from papers discussing resolution criteria of proximate spectral lines. Their application to double stars must have come later, but in each instance I have not been able to identify who was responsible for applying them.

The first paper to deal with diffraction limited resolution was published in the Philosophical Magazine by Lord Rayleigh in February 1874,[11] (Lord Rayleigh's given name was John W. Strutt), under the title, *"On the Manufacture and Theory of Diffraction-gratings."* Rayleigh's resolution criterion is based upon a separation of two spurious discs by an angular distance corresponding to the disc - first interspace ratio, and was given as $r = 0.61\lambda f/R$. Nowadays it is usually quoted as $r = 1.22\ \lambda f/D$, which for a wavelength $\lambda = 550$nm, equates to $r = 138/D$(mm) (where D is the objective aperture in millimetres.)

Much later, the American physicist, C.M. Sparrow,[12] derived a tighter criterion, corresponding to the separation where the intensity dip between the two central discs vanishes (107/D(mm)). Ironically Sparrow actually stated his criterion could be applied to the telescopic resolution of double stars, yet it is almost unknown to astronomical literature!

Dawes' limit falls between these theoretical limits, and none of course apply to unequal pairs, which form the vast bulk of visual binaries. How is it then the Rayleigh and Dawes' resolution limits have become a siné qua non, applied with so little rigour?

The mathematics of diffraction theory is complicated, and most astronomers do not understand it. Fewer yet are capable of applying it to such matters as the resolution of unequal binaries, and are happy to adopt a simple relationship and largely disregard inconvenient niceties. Nevertheless, there is a need of an objective criterion of resolution of double stars that accounts for the intensity difference of the components.

As far as I have been able to ascertain there have only been two concerted efforts (beside my own) to investigate the performance of telescopes on unequal pairs, and only one on the comparative performance of a telescope on equal bright and equal faint pairs, and neither have produced any practical result.

In a short paper published in 'The Observatory' April 1946,[13] the Jesuit double star observer, P.J. Treanor presented, in graphical form, his analysis of data furnished by Lewis. Lewis, a professional double star observer based at Greenwich, was employed to use the 28-inch Grubb refractor. In 1914 [15] he published the results of a selection of the most difficult objects measured by over fifty different observers. These were divided into four groups, the number of pairs on each group being reduced to about five. The mean magnitude and separations for each group were given in a table. These means therefore summarised the observation of some 800 difficult binaries, with a wide selection of instruments and observers.

Treanor plotted the separations, expressed as a function of Dawes' limit, against the magnitude difference. In all, 160 such coordinates were plotted on his diagram. He added four coordinates which represented means adopted by Lewis for nearly equal bright and faint pairs, and unequal and very unequal pairs. He also added a modification of the Rayleigh criterion based on the results of laboratory observations of the diffraction patterns of artificial doubles.

Treanor's modification of the Rayleigh limit was quite straightforward, and it is surprising no one had considered it sooner. He reasoned that resolution occurs if the faint companion falls on a minima of the brighter star, such that the adjacent maxima are not brighter than the faint star itself.

The rationale is sound enough (given one accepts the Rayleigh resolution criterion has objective reality - for further reading on the acuity of vision applied to the resolution of equal and unequal stars refer to [16] *"Optics - The Science of Vision" Ch.Vp205 - 'The Acuity of Vision"* - which deals with Fechner's Law and the Rayleigh criterion) although it is altogether too pessimistic when one considers the case of nearly equal pairs. In the case of pairs of equal brightness, the first minimum fulfils this condition, the adjacent maxima being the Airy disc centre of one component, so that the rule agrees with that of Rayleigh. However, at the Rayleigh limit, and especially where faint pairs are involved, the separation of the two maxima is such that a 2.8 magnitude difference can still be resolved. And where, especially in small telescopes, in perfect seeing, the spurious disc remains steady, the fainter component may be detected when it lies on the neighbouring inner minima rather than the neighbouring outer minima.

Because the Bessel functions for the spurious disc enable the normalised intensity of the central light and rings, and their angular diameters to be calculated, both for unobstructed and obstructed circular apertures, it is a straightforward process to apply Treanor's idea.

The table below lists the disc-maxima-minima ratios for various obstruction ratios (ε), the normalised intensities of the point spread function, $\frac{I}{I_0}$ and the equivalent magnitude difference [6], $.\Delta m = 2.5\lg_{10}\frac{I}{I_0}$.

		disc	disc							
		centre	**edge**		**ring 1**		**ring 2**		**ring 3**	**L.*f* / #**
ε=0	**S/p**	**0**	**0.515**	**1.220**	**1.635**	**2.233**	**2.679**	**3.238**	**3.699**	
I/Io		**1**	**0.5**	**0**	**1/57**	**0**	**1/240**	**0**	**1/625**	
Delta m				**0**	**4.39**	**0**	**5.95**	**0**	**6.99**	
e=0.2		**0**	**0.503**	**1.167**	**1.630**	**2.357**	**2.692**	**3.088**	**3.676**	
I/Io		**1**	**0.5**	**0**	**1/33**	**0**	**1/668**	**0**	**1/269**	
Delta m				**0**	**3.79**	**0**	**7.06**	**0**	**6.07**	
e=0.33		**0**	**0.486**	**1.098**	**1.607**	**2.424**	**2.742**	**3.137**	**3.637**	
I/Io		**1**	**0.5**	**0**	**1/19**	**0**	**1/730**	**0**	**1/480**	
Delta m				**0**	**3.17**	**0**	**7.16**	**0**	**6.7**	
e=0.4		**0**	**0.474**	**1.058**	**1.583**	**2.388**	**2.768**	**3.301**	**3.658**	
I/Io		**1**	**0.5**	**0**	**1/14**	**0**	**1/299**	**0**	**1/1475**	
Dm				**0**	**2.88**	**0**	**6.19**	**0**	**7.92**	

Treanor's modification of Rayleigh's limit formed a tolerable limit to the observations of nearly equal bright pairs, but as a resolution criterion under average observing conditions was much too optimistic for very unequal pairs resolved using large telescopes. Why this was so will be discussed later.

The problem was tackled independently by two American amateur double star observers in the early 1950's. At the 1952 meeting of the mid-states region of the Astronomical League, S.L. O'Byrne [17] explained a prediction formula that he had evolved for his own 2-inch refractor when it was equipped with a x33 eyepiece. From his extensive observations he concluded that the main factor was the difference in magnitude between the fainter component and the magnitude limit of the telescope, rather than the difference between the components. The wider the pair however, the fainter the companion that could be resolved.

O'Byrne's results were emulated by Harold H. Peterson,[18] and his observations of 323 pairs with companions ranging from +4.5mv to +11.5mv, and separations between 1"arc & 100"arc, extracted from William Tyler Olcott's, "Field Book of the Skies," (a gem of a handbook, probably the best ever written) were presented graphically (ref.fig.1). The resultant, resolved and unresolved zone boundary could be described mathematically by the equation:

$$m_2 = L - 2.4 + 1.6\lg_{10}\frac{S}{S_0}$$

where L is the limiting magnitude of the telescope
m2 is the magnitude of the companion
S is the separation in arcsecs & So is the limiting resolution on bright stars for the eyepiece

Peterson gave the limiting magnitude for his 3-inch refractor as +11.1mv. However the limiting magnitude one could realistically expect from a small refractor in a dark sky location is given by: $m_1 = 2.7 + 5\lg_{10} D_{(mm)}$ which when resubstituted into Petersons equation and rearranged for $\frac{S}{S_0}$, gives:

$$\frac{S}{S_0} = \lg_{10}^{-1}\left\{\frac{5}{8}\left[m_1 + 0.7 - 5\lg_{10} D - (m_2 - m_1)\right]\right\}$$

where $\Delta m = m_2 - m_1$

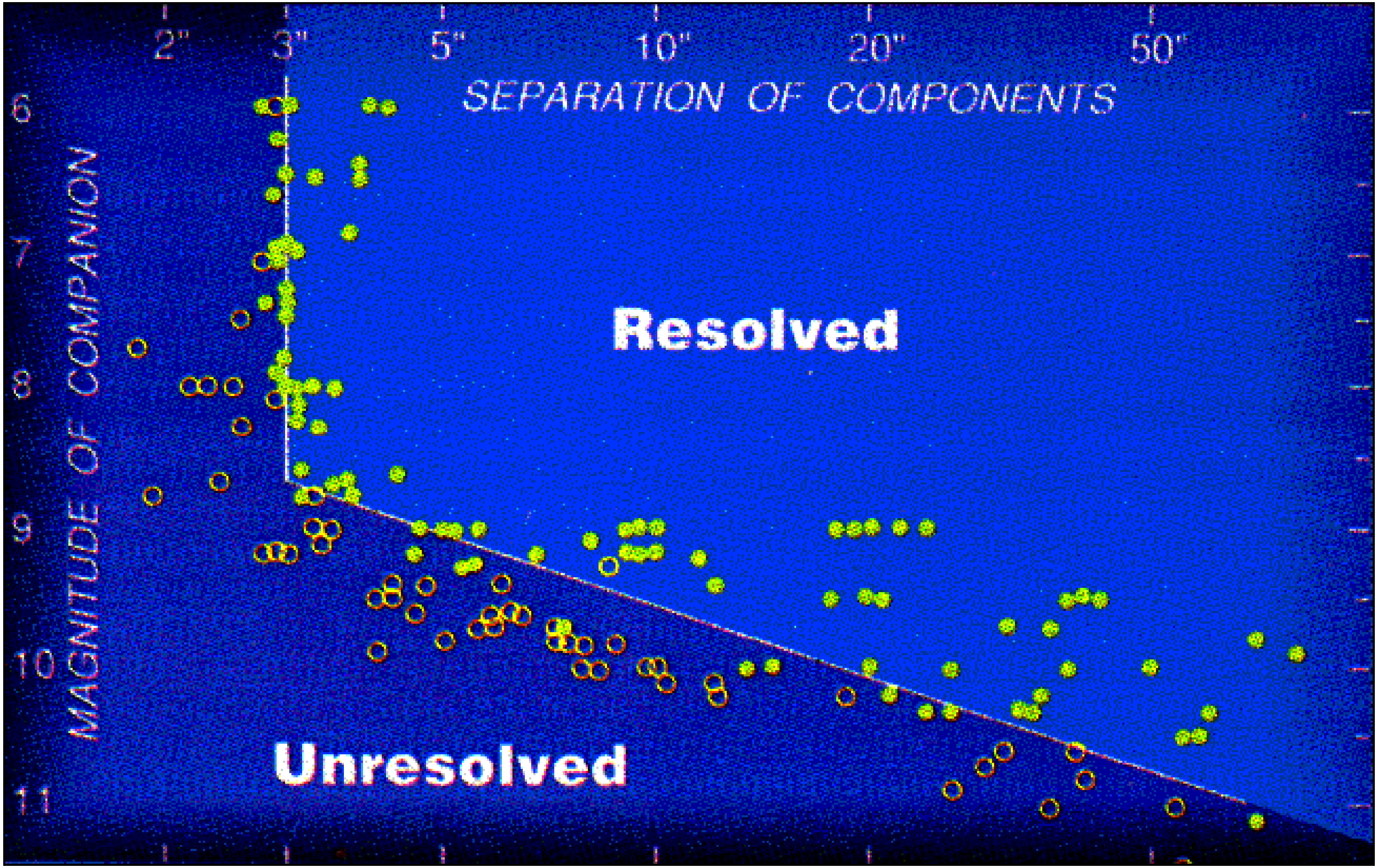

2"
3"
5"
10"
20"
50"
SEPARATION OF COMPONENTS
MAGNITUDE OF COMPANION
6
7
8
9
10
11
Resolved
Unresolved

fig.1

Clearly O'Byrne & Peterson were not measuring the resolution limit of their telescopes. After all, at such low powers the apparent angular separation of pairs near the Rayleigh limit cannot be resolved by the eye. Foveal resolution is approximately 70"arc [19] and for clear recognition, twice that, so the minimum magnification needed to resolve diffraction limited separations is approximately x25 per inch of aperture, the so called Whittaker rule.[20]

Peterson's zone diagram is a measure of the resolution of the eye. The slope cut off, approximately three magnitudes above the telescopic threshold is the point below which scotopic (peripheral) vision becomes dominant.

The conclusions to be drawn from Treanor's & Peterson's results are clear. Large telescopes rarely perform anywhere near their diffraction limit. At low light levels the eye is dependent upon scotopic vision, visual acuity is much reduced, and the Fechner-Weber ratio increases.

Any modification of Dawes' limit predicting the resolution of unequal pairs must be constructed within these considerations. It is no use attempting to include pairs that may only be detected using peripheral vision. Resolution limits must take the departure from diffraction limited pereformance due to seeing into consideration, and be capable of modelling practical performance limits on unequal pairs using unobstructed and obstructed telescopic systems.

A revision of Dawes' limit involves a consideration of the three independent variables, magnitude difference, aperture and separation. The true relationship, of which Dawes' limit will be a special case when $\Delta m = 0$, will be represented on a three axis plot by a hyperbolic interface between resolved and unresolved components. The slope of the interface will be at a maximum when $\Delta m = 0$ and correspond with Dawes' limit. But as magnitude difference Δm increases the slope of the interface will decrease as more aperture is needed to resolve pairs of a given separation.

Data collected into means as a function of Dawes' limit can be plotted onto a two axis graph that will resemble a slice through the three axis graph, and superimposed plots for the different apertures should resemble one another.

The law will be a log.log form:- $\lg S = \lg k - \lg D - \oint \Delta m$

where $\frac{S.D}{k} = \oint m$ is the limiting term ...(corresponding to Dawes' limit)

k is a constant for a particular unit of D

$\oint m$ is a magnitude function

$\oint \Delta m$ is the magnitude difference function

In collecting data to determine the coefficients the magnitude of the primes (m_1) will introduce a selection effect - pairs of a given difference (Δm) will be more easily resolved as D increases. Pairs of a given difference, magnitude of primary a fixed amount above threshold, should be equally resolved in all apertures, therefore pairs have to be binned in magnitude difference and magnitude of comites above threshold.

From such an analysis with a:
3 - inch *f* / 16 refractor, @ X180 (1/4 inch Tolles)
6 - inch *f* / 13.5 Cooke refractor, @ X340 (1/4 inch Tolles)
6 - inch *f* / 16 Maksutov-Cassegrain, Quantum 6, @ X250 (3/8 inch Monocentric)
10-inch *f* / 10.5 Newtonian, by Calver, @ X400 (1/4 inch Tolles)
the empirical coefficients were determined by the method of least squares. (See appendix)

Binaries were selected from Burnham's Celestial Handbook.[21] Only those whose orbits were given in Sky Catalogue 2000 [22] were chosen, and no red pairs, or pairs with red comites were included.

Analysing the means, comprising 94 selected pairs, I have derived by logarithmic regression the following empirical expression (see appendix):

$$\Delta m = 0.1 + 7 \lg_{10} \oint m \qquad \text{-------------------- (i)}$$

the correllation coefficient **r** = 0.9998

rearranging (i):

$$\oint m = \lg_{10}^{-1} \frac{1}{7}(\Delta m - 0.1)$$

but $$S = \oint m \frac{k}{D} \qquad \text{-------------------- (ii)}$$

when **D** is expressed in millimeters, **k** = 120

therefore $$S = \frac{120}{D} \lg_{10}{}^{-1} \frac{1}{7}(\Delta m - \delta m) \qquad \text{-------------------- (iii)}$$

This expression is the statistical median of a family of empirical laws which describe my results. It may be generalised to incorporate any possible pair, observed in any telescope, under any conditions, as follows:

$$\frac{S}{\rho} = \frac{1}{10^{\frac{\delta m}{n'}}} \lg_{10}^{-1} \frac{1}{n}(\Delta m - \delta m)$$

$$i.e.\ \frac{S}{\rho} = 1.033 \lg_{10}^{-1} \frac{1}{n}(\Delta m - 0.1)$$

where $\frac{S}{\rho}$ is a function of Dawes' limit $\left(\frac{116}{D_{(mm)}} = 1\right)$

n is an index defining telescopic performance

(n' is a median = 7)

δm is the least significant magnitude difference (-0.1mv)

Extrapolating my results to Treanor's diagram the bulk of Lewis' means are represented by performance indices n = 4 thru' n = 12. The median lies near n' = 7 and Lewis' means near n = 6. (A table of Lewis' means and my own are included in the appendix).

The performance index is dependent on three factors; seeing, aperture and obstruction ratio. I have tabulated values of n corresponding to each factor - to obtain the performance index, simply add the factors:

APERTURE		**OBSTRUCTION RATIO**		**SEEING**	
Dmm	**nD**	**ε**	**ne**	**SPURIOUS**	**ns**
				DISC RAD.	
<75	**4**	**0**	**4**	**I<0.25p**	**4**
75-150	**3**	**0.1**	**3**	**II>0.25p**	**3**
151-300	**2**	**0.2**	**2**	**III<0.50p**	**2**
301-450	**1**	**0.33**	**1.5**	**IV<1.00p**	**1**
451-600	**0.5**	**0.4**	**1**	**V>1.50p**	**0.5**
>600	**0.5-0.25**	**0.5**	**0.5**		

$$\mathbf{n = n_D + n_e + n_S}$$

N.B. $\alpha\rho$ spurious disc radius in units of $1.22\lambda f/\#$

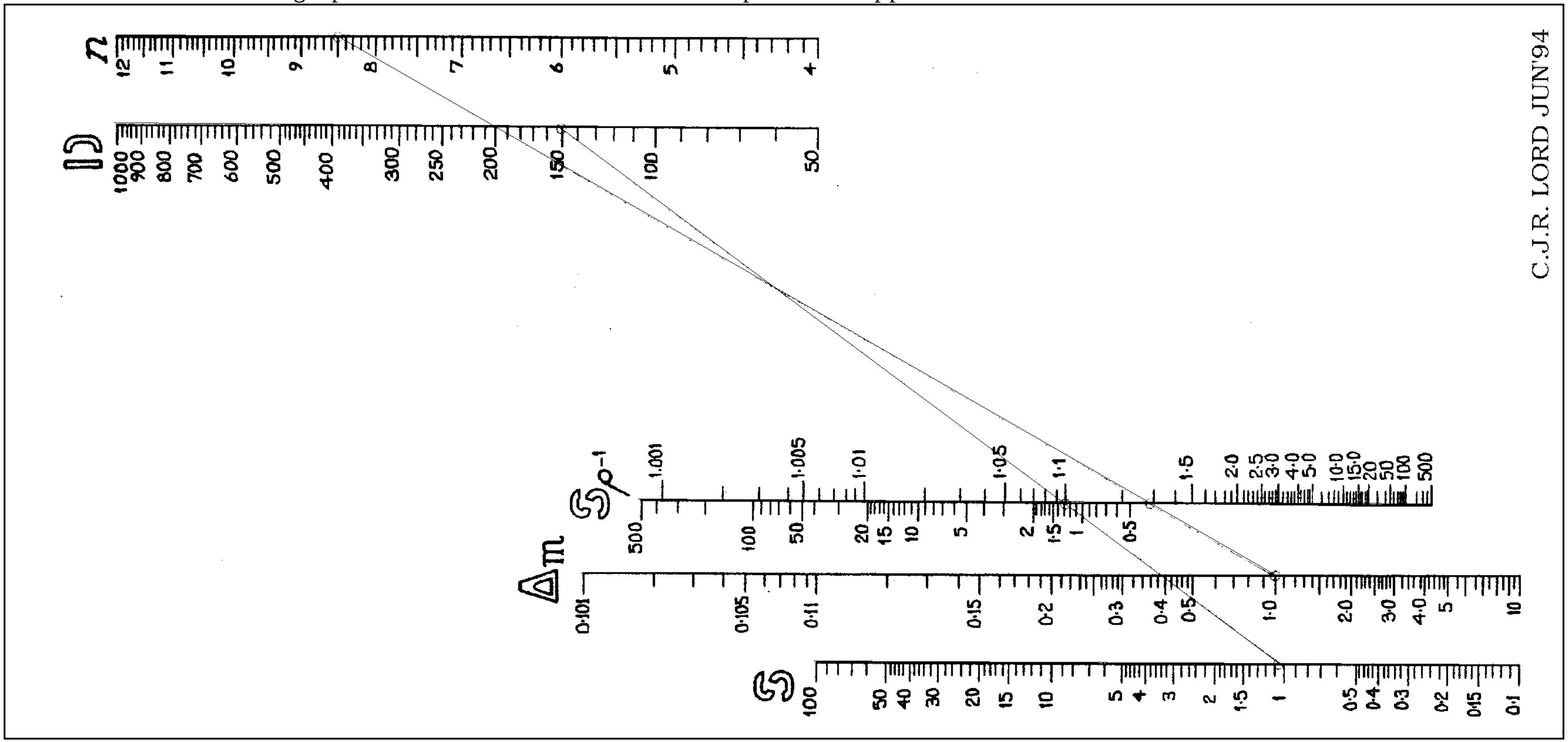
The effect of a central obstruction has only a marginal improvement on the separating power on nearly equal pairs near the Rayleigh limit, as Δm increases, the effect is a reduction in the separating power. (ref. appendix)

For a specific telescope, aperture and obstruction ratio are fixed, therefore the only variable effecting the performance index becomes the seeing. This means a list of seeing dependent performance indices can be made, for a particular diffraction limited telescope.

Example: **150mm MAKSUTOV;** ε **= 0.33** **n = n s + 4 1/2**

SEEING	n
I	8 1/2
II	6 1/2
III-IV	5 1/2
V	5

To facilitate the application of my generalised empirical law I have constructed a nomogram. To enable the interested reader to construct his or her own graph I have included a solution example in the appendix.



To use the nomogram -

(i) determine the appropriate performance index from the table;
(ii) construct an isopleth to the particular value of Δm;
(iii) read off the corresponding resolution function $\frac{S}{\rho}$ using the right hand scale;
(iv) construct an isopleth connecting the telescope aperture and the resolution function $\frac{S}{\rho}$ defined on the left hand scale;
(v) read off the corresponding separation in arcsecs.

The modified Rayleigh limit for $\varepsilon = 0$ is described by the term: $\Delta m = 2.810 + 7.370 \lg_{10}^{-1} \frac{S}{\rho}$ ----------- (see appendix for derivation).

The Rayleigh limit may be applied to unequal pairs below a critical value of Δm corresponding to the intensity difference of the central light and the first ring. When $\frac{S}{\rho} = 1.22\lambda\, f/D$, $\Delta m = 3.44 m_v$ I have mapped the resolution boundary defined by these limits. **'n'** isopleths connect pairs of congruent difficulty. Difficulty increases with the numerical magnitude of **'n'.** Note $\frac{S}{\rho}$ equates to arcsecs when D = 113mm (λ = 550nm). To convert to arcsecs for any other aperture, divide 113 by that aperture in millimetres.

The conclusions one may draw from my analysis of Lewis' results and my own are that, on the whole, small telescopes perform more efficiently than larger ones. Even in rather good observing conditions, large telescopes rarely attain their theoretical limit. There are several reasons for this. Apertures greater than 250mm to 300mm are influenced by seeing to a much greater extent. Residual spherical aberration and a central obstructon will also reduce telescopic performance on unequal binaries. There is also an equivalence between residual spherical aberration and a central obstruction. Conrady[23] showed the Airy disc is modified in a similar fashion in the presence of either a $\frac{1}{3}$ central obstruction or a $\frac{1}{4}\lambda\ OPD$ spherical aberration. Danjon & Couder[24] also pointed out that even slight residual spherical aberration, within the Rayleigh[25] criterion for diffraction limited performance $\frac{1}{4}\lambda\ OPD$ will greatly increase sensitivity to seeing. Reflectors are particularly prone to impairment of resolving power due to thermal lag, and the concomittant departure from diffraction limited performance. The central obstruction exascerpates this departure, and reduces the efficiency in separating unequal pairs.

I would like to thank Richard Berry and David Stoltzmann for supplying the computer generated Point Spread Functions showing the effects of increasing obstruction ratio and the appearance of unequal pairs corresponding to Treanor's modification to Rayleigh's limit, and to Roger Sinnott for permission to reproduce Peterson's diagram. I would also like to express my gratitude to Phil Horrocks for checking my mathematical analysis , reading the proofs and suggesting improvements.

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Appendix:

To construct graphical representation for a range of performance indices; adopt a value of n and determine the Δm scale height corresponding to it.

$$\text{from} \quad \Delta m = 0.1 + n \lg_{10} \oint m$$

$$\frac{\Delta \lg_{10} S}{\Delta_{\Delta m}} = \frac{d \oint m}{d \Delta m} = \frac{1}{n}$$

$$\text{scalar offsets } 90 - \vartheta = 90 - \sin^{-1} \frac{\lg_{10} S_n}{\lg_{10} S_{n_o}} = \cos^{-1} \frac{n_o}{n}$$

e,g, adopt no = 6 using log/log graph paper, minimum 3 x 3 cycles

$$scale\,height = cycle\ length \times \Delta \lg_{10} S_{n=6} \times \Delta m_{10} \times \sin 45^{\circ}$$

$$= cycle\ length \times \frac{1}{n_0} \times \frac{10}{\sqrt{2}}$$

$$= 1.1785 \times cycle\ length$$

$$\Delta m\ scalar\,increment = \frac{1.1785 \times cycle\ length}{10}$$

$$= 0.11785 \times cycle\ length$$

n	$90 - \vartheta$
n = 6	90 - ϑ = 0°
6.5	22.6
7	31.0
7.5	36.9
8	41.4
8.5	45.4
9	48.2
10	53.1
11	56.9
12	60.0

Appendix (cont): Tabulated means of double star observations:

3-inch O.G. Dawes' limit = 1".53

	NEARLY EQUAL PAIRS						UNEQUAL PAIRS					
Star Name	BRIGHT	SEP	S/ρ	FAINT	SEP	S/ρ	Δm $3m_V$	SEP	S/ρ	Δm $5m_V$	SEP	S/ρ
Σ1871	7.0 7.0	2.0	1.31									
Σ1785	7.0 7.5	2.8	1.83									
OΣ288	6.5 7.0	1.5	0.98									
AC1	7.0 7.5	1.7	1.11									
Σ2624	7.0 7.5	1.6	1.05									
MEANS S/ρ = 1.256 = 5.73/D Δm = 0.40												
OΣ279				7.0 9.0	2.2	1.44						
Σ1858				7.0 8.0	2.7	1.76						
OΣ296				7.5 9.0	1.9	1.24						
Σ33				8.0 8.0	2.6	1.70						
Σ179				7.0 8.0	3.6	2.35						
J781				9.5 9.5	2.8	1.83						
MEANS S/ρ = 1.720 = 7.84/D Δm = 0.92												
Σ2665							6.5 9.0	3.3	2.16			
εDRA							4.0 7.5	3.1	2.03			
Σ2694							6.5 10.0	4.0	2.61			
OΣ140							7.0 9.5	2.8	1.83			
β100							7.0 10.5	3.3	2.16			
MEANS S/ρ = 2.158 = 9.84/D Δm = 3.10 n = 9.4												
Ho131										7.5 11.0	4.1	2.68
ηDRA										3.0 8.0	5.3	3.46
Σ2640										6.0 10.0	5.4	3.53
δGEM										3.5 8.0	6.3	4.12
Σ2103										6.0 10.0	5.3	3.46
Σ2142										6.0 10.0	5.2	3.40
MEANS S/ρ = 3.442 = 15.70/D Δm = 4.17 n = 7.8												

MEAN n = 8.7 $\Delta m = -0.25 + 8.7 \lg_{10} S/\rho$ r = 0.9797

Appendix (cont): Tabulated means of double star observations:
6-inch O.G. Dawes' limit = 0".76

	NEARLY EQUAL PAIRS						UNEQUAL PAIRS					
Star Name	BRIGHT	SEP	S/ρ	FAINT	SEP	S/ρ	Δm $3m_v$	SEP	S/ρ	Δm $5m_v$	SEP	S/ρ
ζBOO	4.5 5.0	0.9	1.18									
OΣ410	6.5 7.0	0.8	1.05									
λCYG	5.5 6.5	0.7	0.92									
Σ2438	7.0 7.0	0.7	0.92									
OΣ369	7.0 7.5	0.7	0.92									
MEANS S/ρ = 0.998 = 4.55/D Δm = 0.5												
Σ55				8.0 9.0	2.3	3.03						
OΣ21				7.0 8.0	0.7	0.92						
S141				8.0 8.5	1.6	2.11						
OΣ383				7.0 8.0	0.9	1.18						
OΣ386				7.5 8.0	0.9	1.18						
Ho153				8.0 9.0	0.9	1.18						
MEANS S/ρ = 1.600 = 7.30/D Δm = 0.83												
15BOO							5.5 8.0	1..0	1.32			
δCYG							3.0 6.5	2.2	2.89			
τCYG							4.0 6.5	0.9	1.18			
Σ1709							7.0 10.0	2.4	3.16			
Σ2027							6.0 9.5	2.3	3.03			
MEANS S/ρ = 2.316 = 10.56/D Δm = 3.00 n = 6.3												
OΣ482										5.5 10.0	3.5	4.61
β27										7.0 11.0	3.5	4.61
A2225										7.5 12.0	3.0	3.95
A1418										7.5 11.5	2.9	3.82
Ho161										7.0 11.0	2.9	3.82
51										6.0 12.0	3.0	3.95
MEANS S/ρ = 4.127 = 18.82/D Δm = 4.50 n = 7.3												

MEAN n = 6.5 $\Delta m = 0.54 + 6.5 \lg_{10} S/\rho$ r = 0.99934

all $\Delta m = 0.14 + 7 \lg_{10} S/\rho$ r = 0.963

Appendix (cont): Tabulated means of double star observations:
6-inch MAK. Dawes' limit = 0".76

	NEARLY EQUAL PAIRS						UNEQUAL PAIRS					
Star Name	BRIGHT	SEP	S/ρ	FAINT	SEP	S/ρ	Δm 3m_v	SEP	S/ρ	Δm 5m_v	SEP	S/ρ
γ2AND	5.5 6.0	0.7	0.92									
Σ2049	6.5 7.5	1.0	1.32									
Σ2203	7.5 7.7	0.6	0.79									
Σ2215	6.0 7.5	0.6	0.79									
OΣ338	7.0 7.5	0.7	0.92									
OS215	7.0 7.0	1.1	1.45									
32ORI	5.0 6.5	0.7	0.92									
MEANS	S/ρ = 1.016 = 4.63/D Δm = 0.71											
OΣ298				7.5 7.5	1.0	1.32						
Σ1867				7.5 8.0	1.0	1.32						
OΣ287				7.5 7.5	1.0	1.32						
β398				9.0 9.0	1.8	2.37						
Σ3113				8.5 8.5	0.8	1.05						
β728				8.5 8.5	1.2	1.58						
MEANS	S/ρ = 1.074 = 4.89/D Δm = 0.08											
Σ133							7.0 10.5	3.1	4.08			
Ho502							7.5 10.5	2.4	3.16			
Σ1709							7.0 10.0	2.4	3.16			
Ho399							7.5 10.0	3.2	4.21			
Σ2022							6.0 9.5	2.3	3.03			
OΣ373							7.5 10.0	1.7	2.24			
β145							7.0 9.5	0.8	1.05			
MEANS	S/ρ = 2.990 = 13.63/D Δm = 2.79 n = 5.8											
β189										7.0 11.5	4.2	5.53
Hu1225										7.5 12.0	3.0	3.95
β1048										6.0 10.5	2.0	2.63
OΣ111										6.0 10.0	2.6	3.42
34PEG										6.0 11.5	3.5	4.61
α_2CAP										3.6 10.4	6.6	8.68
MEANS	S/ρ = 4.028 = 18.37/D Δm = 4.4 n = 7.3											
MEAN	n = 5.7 Δm = 0.565 + 5.7$\lg_{10}$S/ρ r = 0.9706											

Appendix (cont): Tabulated means of double star observations:

10-inch SPEC Dawes' limit = 0".46

Star Name	NEARLY EQUAL PAIRS BRIGHT	SEP	S/ρ	FAINT	SEP	S/ρ	UNEQUAL PAIRS Δm $3m_v$	SEP	S/ρ	Δm $5m_v$	SEP	S/ρ
44BOO	5.5 6.0	0.5	1.09									
A1110	7.5 7.5	0.5	1.09									
Σ1863	7.5 7.5	0.5	1.09									
36AND	6.0 6.5	0.5	1.09									
13PEG	5.5 7.0	0.4	0.87									
Ho296	6.0 6.5	0.4	0.87									
MEANS S/ρ = 1.017 = 4.64/D Δm = 0.5												
OΣ276				7.5 8.5	0.5	1.09						
OΣ277				8.0 8.0	0.4	0.87						
Hn60				9.0 9.5	0.8	1.74						
AC14				8.0 9.0	0.8	1.74						
Ho165				8.0 8.0	0.5	1.09						
Ho179				8.0 9.0	0.9	1.96						
MEANS S/ρ = 1.415 = 6.45/D Δm = 0.60												
β1166							8.0 11.0	2.7	5.87			
OΣ1							7.5 10.0	1.6	3.48			
β1227							7.5 11.5	2.9	6.30			
Σ277							7.5 11.0	3.1	6.74			
Σ2977							7.0 10.5	1.9	4.13			
MEANS S/ρ = 5.304 = 24.19/D Δm = 3.30 n = 4.5												
β1271										7.0 12.0	2.4	5.22
β1310										7.0 12.0	3.7	8.04
A1253										7.5 12.5	3.0	6.52
OΣ512										6.5 11.0	3.0	6.52
β1140										7.0 12.0	4.1	8.91
Ho254										6.5 12.5	2.5	5.43
oPER										4.0 8.5	1.0	2.17
OΣ76										7.5 12.0	3.9	8.48
MEANS S/ρ= 6.411 = 29.24/D Δm = 4.90 n = 6.1												

MEAN n = 4.9

$\Delta m = 0.40 + 4.9 \lg_{10} S/\rho$ $\quad r = 0.9628$

all $\Delta m = 0.14 + 5.2 \lg_{10} S/\rho$ $\quad r = 0.9672$

Appendix (cont): Tabulated means of double star observations:
LEWIS' MEANS ABSTRACTED FROM :-
'The OBSERVATORY' Oct.1914 No.479 p378

		S/ρ		S/ρ		S/ρ		S/ρ	
OBSERVER	**APERTURE**	**BRIGHT**	**Δm**	**FAINT**	**Δm**	**UNEQUAL**	**Δm**	**VERY UNEQUAL**	**Δm**
JACOB	4 ".0	1.40	0.5	1.93	0.6	2.37	3.3	3.77	4.8
COMAS SOLA	6.0	1.05	0.5	2.11	0.3	2.63	2.6	6.45	4.4
MAW	6.0	1.32	0.8	2.24	0.8	2.76	2.0	3.16	4.0
BURNHAM	6.0	0.53	0.4	1.18	1.2	2.50	3.9	3.95	6.3
JACOB	6.3	1.66	0.9	2.49	0.6	2.49	3.3	4.70	5.6
DAWES	7.0	1.23	0.9	1.38	0.3	2.92	3.4	2.92	4.8
KNOTT	7.3	1.12	0.3	1.92	1.3	1.92	3.7	7.84	6.1
DEMBOWSKI	7.4	0.65	0.3	1.14	0.2	1.95	3.1	3.73	5.1
MAW	8.0	1.23	0.7	2.46	0.6	2.46	2.7	6.14	4.3
COLEMAN	8.0	1.23	0.9	1.40	0.2	3.68	3.1	7.02	4.0
ENGELMANN	8.5	1.12	0.5	1.86	0.6	3.17	3.8		
BURNHAM	9.4	0.72	0.4	1.24	0.5	2.27	3.0		
FARMAN	9.5	0.79	0.7	2.50	0.7	2.71	3.1	4.38	4.8
DUNÊR	9.6	0.82	1.0	2.74	0.4	3.37	2.8	8.42	5.3
W.STRUVE(S)	9.6	0.74	0.6	1.05	0.3	1.89	2.8	3.37	4.4
SECCHI	9.6	0.80	0.3	1.05	0.5	1.68	2.8	4.21	4.1
LEAVENWORTH	10.5	0.83	0.3	1.61	0.8	2.53	3.1	8.75	4.5
HOWE	11.0	1.33	2.4	2.17	1.1	2.65	3.0	6.51	4.9
BURNHAM	12.0	1.01	0.8	1.21	0.4	2.89	3.4	5.07	5.9
HUSSEY	12.0	0.82	0.9	1.21	0.3	2.17	2.4		
BIGOURDAN	12.0	1.06	0.8	1.69	0.4	2.89	2.6	4.34	5.2
BIESBROECK	14.8	0.94	0.6	1.62	0.5	2.92	3.0	3.89	4.8
PERROTIN	15.0	1.35	1.1	2.30	0.7	4.28	3.1	11.51	5.0
H.STRUVE	15.0	1.97	0.6	3.62	2.9	7.57	4.1		
DOOLITTLE	18.0	1.18	0.0	3.95	0.5	4.74	4.4	8.68	5.6
DOOLITTLE	18.0	1.26	0.4	1.97	0.4	3.95	3.5	6.71	5.3
HOUGH	18.5	1.18	0.0	3.65	0.4	4.46	4.4	7.71	5.6
BURNHAM	18.5	1.62	0.2	2.43	0.4	3.25	2.8	4.87	5.9
SCHIAPARELLI	19.2	0.80	0.5	1.68	0.3	1.68	3.4	5.05	5.3
SEE	24.0	1.37	0.4	2.63	0.3	4.74	3.5	7.37	5.2

Appendix (cont): Tabulated means of double star observations:
LEWIS' MEANS ABSTRACTED FROM
'The OBSERVATORY' Oct.1914 No.479 p378

		S/ρ		**S/ρ**		**S/ρ**		**S/ρ**	
OBSERVER	**APERTURE**	**BRIGHT**	**Δm**	**FAINT**	**Δm**	**UNEQUAL**	**Δm**	**VERY UNEQUAL**	**Δm**
HALE	26".0	1.60	0.5	4.56	1.8	6.84	2.6	10.26	4.6
WILSON	26.0	3.42	0.9	3.42	0.0	6.84	2.9	23.38	4.8
OLIVER	26.0	3.42	0.9	3.42	0.5	5.70	2.9		
NEFF	26.0	4.56	0.7	4.56	0.7	5.70	3.2		
LEWIS	28.0	0.61	0.7	1.54	0.3	3.07	3.6	6.14	6.6
BOWYER	28.0	1.04	0.2	1.90	0.5	5.53	4.1	14.74	5.9
H.STRUVE	30.0	1.71	0.7	4.61	0.9	5.92	3.7	10.53	6.4
BURNHAM	36.0	1.18	0.4	1.82	0.2	7.11	4.7	7.11	6.6
HUSSEY	36.0	1.18	0.2	1.58	0.3	3.95	2.9	18.16	6.2
AITKEN	36.0	1.42	0.3	1.58	0.2	3.95	3.4	19.74	6.2
HUSSEY (OS)	36.0	1.26	0.8	3.16	0.9	4.74	2.6		
HUSSEY	36.0	0.71	0.2						
AITKEN	36.0	0.63	0.2	1.89	0.4				
MEANS		1.283	0.59	2.210	0.52	3.583	3.21	7.661	5.2
				n = 5.8		n = 5.9			

$\Delta m = -0.06 + 5.937 \lg_{10} S/\rho$ r = 0.99997   N = 159

LORD	3".0	1.26	0.4	1.72	0.9	2.16	3.1	3.44	4.2
LORD	6.0	1.00	0.5	1.61	0.8	2.32	3.0	4.12	4.5
LORD	6.0	1.01	0.7	1.08	0.1	2.99	2.9	4.03	4.6
LORD	10.0	1.01	0.5	1.43	0.6	5.31	3.3	6.40	5.0
MEANS		1.07	0.53	1.46	0.6	3.20	3.1	4.50	4.58

$\Delta m = -0.06 + 6.7 \lg_{10} S/\rho$ r = 0.9998   N = 94
n = 6.7
ADOPT $\Delta m = 0.1 + 7 \lg_{10} S/\rho$
MEAN $\Delta m = 0.007$

Appendix (cont): Modification of the Rayleigh limit:

Treanor's modification of the Rayleigh limit may be described mathematically by evaluating the maxima of the point spread function for an infinite conjugate and deriving the relationship between the angular radii and normalised intensities.

The point spread function maxima, radius (kr), for an unobstructed circular aperture, is given by the Bessel function of the first kind J2(kr) = 0.

by recursion: J2(kr) = (2 / kr)J1(kr) - Jo(kr)

Jo(kr) = (2 / kr)J1(kr)

The general solution to the Bessel function, order p, may be reduced to order zero:
when p = 0

from
$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k+p}$$

& when p≥ 1,2,3,........n

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k!)^2} \left(\frac{x}{2}\right)^{2k+n}$$

when p = 0 the general solution reduces to:

$$J_0(x) = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

which may be solved as a second order kind, order zero:

$$Y_0(x) = -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots\dots\dots + \frac{1}{k}\right) \left(\frac{x}{2}\right)^{2k} + \frac{2}{\pi} J_0(x) \left[\ln\frac{x}{2} + \gamma\right]$$

γ (Euler's constant) = $1 + \sum \left(\frac{1}{n} + \ln(n-1)n^{-1}\right)$

γ = 0.57721566

at maxima: $\frac{2}{\pi} J_0(x)\left[\ln\frac{x}{2} + \gamma\right] \Rightarrow 1$

from which it follows that all possible values of (kr) ε= 0 at maxima are related logarithmically. The normalised intensity as a function of (kr) in the point spread function, for an unobstructed circular aperture is given by:

$$\frac{I_{(r)}}{I_0} = \left[\frac{2 J_1(kr)}{kr}\right]^2$$

and the magnitude equivalent by:

$$\Delta m = 2.5 \lg_{10}\left(\frac{I_0}{I_{(r)}}\right)$$

values of (kr) at successive maxima may be calculated using Newton-Raphson iteration:

$$kr_{n+1} = kr_n - \frac{f(kr)}{f'(kr)}$$

at maxima: $f(kr) = -J_2(kr) = J_0(kr) - \frac{2}{(kr)} J_1(kr)$

& $f'(kr) = \left(\frac{4}{k^2 r^2} - 1\right) J_1(kr) - \left(\frac{2}{kr}\right) J_0(kr)$

computing the appropriate values of $J_o(kr)$ & $J_1(kr)$ from the polynomial approximations, the expression for Newton-Raphson iteration becomes:

$$kr_{n+1} = kr - \frac{\left\{J_0(kr) - \frac{2}{kr} J_1(kr)\right\}}{\left\{\left(\frac{4}{k^2 r^2} - 1\right) J_1(kr) - \frac{2}{kr} J_0(kr)\right\}}$$

and quickly converges to the roots of f(kr), namely the maxima of the PSF. Evaluating the PSF(kr_n) after convergence produces the normalised peak intensity $I_{(r)}/I_o$.

(The zeros of the J_o & J_1 Bessel functions are tabulated in Abramowitz & Stegun's "Handbook of Mathematical Functions" - Dover 1964)

I have tabulated the first five maxima below:

DISC FEATURE	$(kr)^{\alpha}$	$\frac{I_{(r)}}{I_0}$
CORE	0	1.0
RING 1	1.634719π	0.017498
RING 2	2.679292π	0.004158
RING 3	3.698710π	0.001601
RING 4	4.709698π	0.000779
RING 5	5.716788π	0.000437

$N.B.\ {}^{\alpha}k = \pi.\lambda.\, f/no.$

The relationship between the resultant values for S/ρ & Δm may be expressed by:

$$\Delta m = A + B\lg_{10}\frac{S}{\rho}$$

$$where \quad \Delta m = 2.5\lg_{10}\frac{I_0}{I}$$

$$\& \quad \frac{S}{\rho} = \frac{k}{\pi}$$

The coefficients A & B being determined by the method of least squares.

$\frac{S}{\rho}$	Δm	
1.634719	4.392 528 969	
2.679292	5.952 788 788	A = 2.809 891 655
3.698710	6.989 021 670	B = 7.370 208 414
4.709698	7.771 156 356	r = 0.999 982 088
5.716788	8.398 796 408	

DEFINED BOUNDARY to RESOLVED UNEQUAL BINARIES

Using the relationship which defines the theoretical limit and combining it with the empirical relationship it now becomes possible to map corresponding values of **n**, which is defined as a performance index.
The higher the value of **n** (always a positive number: $1 < n < \infty$), the more difficult the pair are to resolve.

CRITICAL POINTS:

The resolution limit is defined where $\mathbf{n} \Longrightarrow \infty$

The corresponding values of Δ**m** & **S/**ρ become: $S/\rho = 1.033$, $\Delta m = 2.915$

At the Rayleigh limit, where $S = 1.219670 l.\lambda.f/\#$ $\Delta m = 3.446$, $n = 46.494$

Using these relationships and critical points I have mapped the corresponding performance indices to Δm, $\frac{S}{\rho}$ coordinates. The isopleths connect Δm & $\frac{S}{\rho}$ values of congruent difficulty. It may be seen that within the Rayleigh limit, the most difficult pairs require a performance index n = 46.5. Given that from practical experience the highest working value of $n \le 12$, it is evident that there is an area bounded between $12 \le n \le 46.5$ in which resolution is theoretically possible, but apparently too difficult for the average human eye. There exists therefore the possibility that more sensitive high resolution detectors (e.g. CCD) may be able to exploit this niché.

ENHANCEMENT of RESOLUTION of EQUAL BINARIES due to a CENTRAL OBSTRUCTION

Resolution of equal binaries $(\Delta m \leq 0.1)$ may be marginally improved by introducing a central obstruction.

ε	$\frac{S}{\rho}\left(\lambda \, {}^{f}\!/\!_{D}\right)$	$S = \frac{n}{D_{(mm)}}\ arcsecs$
0.0	1.219670	138
0.1	1.205060	136.7
0.2	1.166507	132.2
0.3	1.114518	126.4
0.4	1.057666	120
0.5	1.000906	113.5
0.6	0.946633	107.4

loss of energy from the disc to the rings imposes a practical limit of about ε=0.6.

To an approximation of one decimal place when ε=0.0 to 0.3

$$S = \frac{\alpha\left(\left(1-\varepsilon^{2}\right)^{-2}\right)^{\beta}.\lambda}{D.\sin 1''} \quad \text{where} \quad \begin{aligned} \alpha &= 1.216917868 \\ \beta &= -0.47436601 \\ r &= -0.99789911 \end{aligned}$$

and when ε=0.3 to 0.6

$$S = \frac{(a+b.\varepsilon).\lambda}{D.\sin 1''} \quad \text{where} \quad \begin{aligned} a &= 1.2786785 \\ b &= -0.553655 \\ r &= -0.99977439 \end{aligned}$$

==